ISRAEL JOURNAL OF MATHEMATICS 154 (2006), 209–220

# A COMPLETE C-SPACE WHOSE SQUARE IS STRONGLY INFINITE-DIMENSIONAL

BY

JAN VAN MILL

*Faculteit Exacte Wetenschappen, Afdeling Wiskunde, Vrije Universiteit De Boelelaan 1081A, 1081 HV Amsterdam, The Netherlands*
*e-mail: vanmill@cs.vu.nl*

AND

ROMAN POL*

*Institute of Mathematics, University of Warsaw Banacha 2, 02-197 Warszawa, Poland*
*e-mail: pol@mimuw.edu.pl*

ABSTRACT

We construct a hereditarily disconnected, complete C-space whose square is strongly infinite-dimensional, and a totally disconnected C-space which is not countable-dimensional (this space is not complete).

## 1. Introduction

All our spaces will be separable and metrizable. Our terminology follows Engelking [3], Kuratowski [8] and van Mill [9]. A space $X$ is a **C-space** if for every sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of open covers of $X$ there are pairwise disjoint open families $\mathcal{B}_1, \mathcal{B}_2, \ldots$ in $X$ such that $\mathcal{B}_i$ refines $\mathcal{A}_i$ for every $i$, and $\bigcup_{i=1}^{\infty} \mathcal{B}_i$ covers $X$. The **weakly infinite-dimensional spaces** are the ones that satisfy the weaker condition that these refinements $\mathcal{B}_i$ exist in case the covers $\mathcal{A}_i$ all consist of at most two elements (no examples distinguishing these properties are known). Any strongly infinite-dimensional space (i.e., a space which is

* The second-named author would like to express his gratitude to the Vrije Universiteit for its hospitality. This author was partially supported by grant 2 PO3A 017 24.
Received September 28, 2004 and in revised form April 2, 2005

not weakly infinite-dimensional) has an infinite essential family — a sequence $\{(A_i, B_i) : i \in \mathbb{N}\}$ of pairs of disjoint closed sets such that if $L_i$ is an arbitrary partition between $A_i$ and $B_i$ for every $i$, then $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. A space is **countable-dimensional** if it is a countable union of zero-dimensional sets. Such spaces are C-spaces.

The product of two C-spaces may be strongly infinite-dimensional; cf. [11], [12]. However, the constructions in the literature illustrating this phenomenon are based on transfinite induction, producing spaces that are not absolutely Borel. One of the main results of this paper is the following:

*Example 1.1:* There is a complete C-space whose square is strongly infinite-dimensional.

The complete C-space we construct is also hereditarily disconnected and, not being countable-dimensional, provides a partial answer to a question asked in [5]. We did not succeed in producing a totally disconnected such space (which would answer the question completely). However, giving up all good descriptive properties, we obtained the following:

*Example 1.2:* There is a totally disconnected C-space which is not countable-dimensional.

The main construction in this paper is based on an approach developed in the paper [10] concerning "splintered spaces" (although this notion will not be explicitly mentioned here), combined with Bing spaces, and some standard methods producing uncountable-dimensional C-spaces.

We collect in §2 some auxiliary results needed for our main construction, which will be presented in §3. The spaces in Examples 1 and 2 are described in §§4 and 5, respectively.

ACKNOWLEDGEMENT: We are indebted to the referee for comments which improved the exposition.

## 2. Auxiliary results

In our constructions, we shall deal with mappings $p$ between spaces $E$ and $F$, such that

(1) $p: E \to F$ is a perfect surjection,

(2) $F$ is complete and zero-dimensional.

In this section we shall make several preliminary observations that will be used later.

LEMMA 2.1: *Let $p$: $E \to F$ be a continuous closed surjection, where $F$ is zero-dimensional. If $A$ is an $F_\sigma$-subset of $E$ such that $p(A) = F$, then there is a countable collection $\mathcal{B}$ of closed subsets of $E$ such that*

(i) $\bigcup \mathcal{B} = A$,

(ii) *the collection $\{p(B) : B \in \mathcal{B}\}$ is pairwise disjoint.*

*Proof:* Let us write $A$ as $\bigcup_{i=1}^{\infty} A_i$, where each $A_i$ is closed in $E$. Put $A_0 = \emptyset$ and fix $i \geq 0$ for a moment. The sets $p(A_0), \dots, p(A_{i+1})$ are closed in $F$. Since $F$ is zero-dimensional, there is a countable pairwise disjoint family $\mathcal{C}_i$ of closed subsets of $F$ with $\bigcup \mathcal{C}_i = p(A_{i+1}) \setminus \bigcup_{j=0}^{i} p(A_j)$. Then

$$\mathcal{B} = \{p^{-1}(C) \cap A_{i+1} : C \in \mathcal{C}_i, i \geq 0\}$$

is clearly as required. ∎

LEMMA 2.2: *Let $p$: $E \to F$ be a continuous closed surjection for which there exists a countable collection $\mathcal{A}$ of closed subsets of $E$ such that the collection $\{p(A) : A \in \mathcal{A}\}$ is pairwise disjoint and covers $F$. If $S(A)$ is a $G_\delta$-subset of $A$, for every $A \in \mathcal{A}$, then $\bigcup_{A \in \mathcal{A}} S(A)$ is a $G_\delta$-subset of $E$.*

*Proof:* For every $A \in \mathcal{A}$ let $B(A) = p^{-1}(p(A))$. Then $B(A)$ is closed since $p$ is closed and continuous. In addition, $B(A) \setminus S(A)$ is an $F_\sigma$-subset of $E$. Hence

$$E \setminus \bigcup_{A \in \mathcal{A}} S(A) = \bigcup_{A \in \mathcal{A}} B(A) \setminus S(A)$$

is an $F_\sigma$-subset of $E$. ∎

Along with the perfect surjection (1), we shall consider the associated monotone-light factorization

(3) $p = u \circ v$, $v$: $E \to F^*$, $u$: $F^* \to F$,

where

(4) $v$ is monotone and $u$ is light;

cf. [6, 3-37]. Let us recall that both maps $u$ and $v$ are perfect, $v$ has connected fibers and the fibers of $u$ are zero-dimensional. In particular, since $\dim F = 0$,

(5) $\dim F^* = 0$.

(This can be verified by a straightforward direct computation. Alternatively, apply [9, Theorem 3.6.10].)

LEMMA 2.3: *Let $p: E \to F$ be a perfect surjection, where $F$ is zero-dimensional. Then for every $\varepsilon > 0$ the set $A(\varepsilon)$ of all points $x \in E$, such that the component of $x$ in $E$ has diameter at least $\varepsilon$, is closed in $E$.*

*Proof:* Considering the factorization (3), (4), we have $A(\varepsilon) = v^{-1}(L)$, where $L$ is the set of points in $F^*$ with fibers of diameter at least $\varepsilon$. Since $L$ is closed, so is $A(\varepsilon)$. ∎

*Remark 2.4:* Let us recall that a function $f: X \to Y$ is of the first Baire class if and only if for every open subset $U$ of $Y$ we have that $f^{-1}(U)$ is an $F_\sigma$-subset of $X$; cf. [7, p. 192].

Let $p: E \to F$ be a perfect surjection. Then by a selection theorem due to Kuratowski and Ryll-Nardzewski [8, §43, IX], there is a first Baire class function $f: F \to E$ with $p \circ f(t) = t$, for $t \in F$.

Let us note that $S = f(F)$ is a $G_\delta$-set intersecting each fiber of $p$ in precisely one point; cf. [2, p. 144, Exercise 9a]. Indeed, if $B_1, B_2, \dots$ are the closures of the elements of a countable base in $E$, then

$$E \setminus S = \bigcup_{i=1}^{\infty} (B_i \cap p^{-1}(F \setminus f^{-1}(B_i))).$$

The following result, which is an essential element in our main construction, is based on a reasoning in the proof of [10, Lemma 3.5].

PROPOSITION 2.5: *Let $p: E \to F$ be a continuous map onto a complete zero-dimensional space $F$. Let $f: F \to E$ be a first Baire class function with $p \circ f(t) = t$ for every $t \in F$, and let $S = f(F)$. Then for each $\varepsilon > 0$ there is a countable collection $\mathcal{S}$ of closed subsets in $E$ such that*

(i) $\mathrm{mesh}(\mathcal{S}) \le \varepsilon$,
(ii) $p(A) \cap p(B) = \emptyset$ *for all distinct* $A, B \in \mathcal{S}$,
(iii) $S \subseteq \bigcup \mathcal{S}$.

*Proof:* Fix $\varepsilon > 0$. Let $A$ be a nonempty closed subset of $F$. We claim that there is a nonempty relatively clopen subset $V$ of $A$ such that $\operatorname{diam} f(V) < \varepsilon$. Indeed, since $f$ is of the first Baire class, there is a point $x \in A$ at which $f{\restriction}A$ is continuous; cf. [8, §34.VII]. Given a neighborhood $U$ of $f(x)$ in $E$ of diameter less than $\varepsilon$, there is a clopen neighborhood $V$ of $x$ in $F$ such that $f(V \cap A) \subseteq U$. Then $V$ is clearly as required.

Using this, we construct by transfinite induction on $\alpha < \omega_1$ closed sets $W_\alpha$ in $F$ such that

(6) if $F \setminus \bigcup_{\beta<\alpha} W_\beta \neq \emptyset$, then $W_\alpha$ is a nonempty relatively clopen subset of $F \setminus \bigcup_{\beta<\alpha} W_\beta$,

(7) $\operatorname{diam} f(W_\alpha) < \varepsilon$.

Observe that by (6) we have that $\bigcup_{\beta\leq\alpha} W_\beta$ is open in $F$ for every $\alpha < \omega_1$. So there is an ordinal $\xi < \omega_1$ such that $F = \bigcup_{\alpha<\xi} W_\alpha$. Now for every $\alpha < \xi$, let $A_\alpha$ be the closure of $f(W_\alpha)$ in $S$. Then $A_\alpha$ is contained in $p^{-1}(W_\alpha)$; therefore the collection $\mathcal{S} = \{A_\alpha : \alpha < \xi\}$ is pairwise disjoint and hence as required. ∎

PROPOSITION 2.6: *Let $p$: $E \to F$, $f$: $F \to E$ and $S = f(F)$ be as in Proposition 2.5 and, in addition, let $p$ be perfect. Then the union $\mathcal{C}(S)$ of the continua in $E$ that intersect $S$ is a $G_\delta$-subset of $E$.*

*Proof:* Let us begin with the observation that for any closed $A$ in $E$, the union $A^*$ of the continua in $E$ that intersect $A$ is closed. This follows readily from the monotone-light factorization (3), (4), as $A^* = v^{-1}(v(A))$.

Now, for every $n$, let $\mathcal{S}_n$ be the collection described in Proposition 2.5 for $\varepsilon = 1/n$, and let $\mathcal{C}_n(S)$ be the union of the continua in $E$ that intersect $\bigcup \mathcal{S}_n$. Then $\mathcal{C}_n(S)$ is the union of the sets $A^*$ with $A \in \mathcal{S}_n$. Using the fact that $F$ is zero-dimensional, we get $p(A^*) = p(A)$ for every $A \in \mathcal{S}_n$. Since the collection $\{p(A) : A \in \mathcal{S}_n\}$ is a partition of $F$, it follows that

$$E \setminus \mathcal{C}_n(S) = \bigcup_{A\in\mathcal{S}_n} (p^{-1}(p(A)) \setminus A^*).$$

So we are done once we establish the following:

$$(*) \qquad \mathcal{C}(S) = \bigcap_{n=1}^{\infty} \mathcal{C}_n(S).$$

Indeed, take an arbitrary $x \in \bigcap_{n=1}^{\infty} \mathcal{C}_n(S)$, and let $t = p(x)$. Then for each $n$ there are $S_n \in \mathcal{S}_n$ and a continuum $C_n$ in $E$ such that $x \in C_n$ and $C_n \cap S_n \neq \emptyset$. Since $C_n \subseteq p^{-1}(t)$ it follows that $S_n \cap p^{-1}(t) \neq \emptyset$; hence $t \in p(S_n)$. The collection $\mathcal{S}_n$ is pairwise disjoint and hence $f(t) \in S_n$. So we conclude that $\varrho(f(t), C_n) < 1/n$. A subsequence of the sequence $C_1, C_2, \ldots$ converges to a continuum $C \subseteq p^{-1}(t)$ which contains both $x$ and $f(t)$. Hence $x \in \mathcal{C}(S)$. The reverse inclusion being trivial, this proves $(*)$. ∎

Another important element of our main construction is described in the following:

PROPOSITION 2.7: *Let $p$: $E \to F$, $f$: $F \to E$ and $S = f(F)$ be as in Proposition 2.6. In addition, let $W$ be a $G_\delta$-subset of $E$ with $W \cap p^{-1}(t)$ finite for every $t \in F$. Then there is $G_\delta$-subset $T$ of $E$ such that:*

(i) *$T \cap p^{-1}(t)$ is a singleton for every $t \in F$, say $T \cap p^{-1}(t) = \{\xi(t)\}$,*
(ii) *for all $t \in T$, there is a (possibly degenerate) subcontinuum of $p^{-1}(t)$ that contains both $f(t)$ and $\xi(t)$,*
(iii) *if the component of $f(t)$ in $E$ is nontrivial, then $\xi(t) \notin S \cup W$,*
(iv) *$T \cap S$ is zero-dimensional.*

*Proof:* Let us consider the monotone-light factorization (3), (4), and put $D = v(S)$. Then $v^{-1}(D) = \mathcal{C}(S)$ is the set introduced in Proposition 2.6. Hence the restriction $\pi$ of $v$ to $\mathcal{C}(S)$ is a perfect monotone map onto the zero-dimensional space $D$. Clearly, $\pi(S) = D$, and for each $s \in S$, the component of $E$ containing the point $s$ coincides with the fiber $\pi^{-1}(\pi(s))$. In addition, $D$ is a $G_\delta$-subset of $F^*$, since $E \setminus \mathcal{C}(S)$ is an $F_\sigma$-set by Proposition 2.6 and $v$ takes closed sets to closed sets.

Let

$$D_0 = \{d \in D : \pi^{-1}(d) \text{ is a singleton}\},$$

and let

$$L = \pi^{-1}(D \setminus D_0) \setminus (S \cup W).$$

Observe that $D_0$ is a $G_\delta$-subset of $D$, $\pi$ being perfect. So $L$ is an $F_\sigma$-subset of $\mathcal{C}(S)$, and since $\pi^{-1}(d)$ for $d \in D \setminus D_0$ is a nontrivial continuum, $\pi(L) = D \setminus D_0$. Since $D$ is zero-dimensional, there are by Lemma 2.1 closed sets $L_1, L_2, \ldots$ in $\pi^{-1}(D \setminus D_0)$ such that $L = \bigcup_{i=1}^{\infty} L_i$, and the sets

$$D_i = \pi(L_i), \quad i = 1, 2, \ldots$$

are pairwise disjoint (and cover $D \setminus D_0$ since $\pi(L) = D \setminus D_0$). The map $\pi \restriction L_i$: $L_i \to D_i$ is perfect; therefore for every $i$ we may choose a $G_\delta$-subset $T_i$ of $L_i$ intersecting each fiber of $\pi \restriction L_i$ in exactly one point; cf. Remark 2.4. It follows by Lemma 2.2 that

$$\pi^{-1}(D \setminus D_0) \setminus \bigcup_{i=1}^{\infty} T_i$$

is an $F_\sigma$-subset of $\pi^{-1}(D \setminus D_0)$ and hence of $\mathcal{C}(S)$. As a consequence,

$$T = \pi^{-1}(D_0) \cup \bigcup_{i=1}^{\infty} T_i$$

is a $G_\delta$-subset of $\mathcal{C}(S)$. Then $T$ satisfies (i) – (iii). To check (iv), let us notice that $T \cap S = \pi^{-1}(D_0)$ and since the fibers $\pi^{-1}(d)$, $d \in D_0$, are singletons, $T \cap S$ is homeomorphic to $D_0$. ∎

We shall close this section with a description of a mapping which will be the starting point of our main construction, given in the next section.

PROPOSITION 2.8: *There are a continuous surjection $p$: $M \to \mathbb{K}$ from a compact space onto the Cantor set $\mathbb{K}$ and a first Baire class function $f$: $\mathbb{K} \to M$ such that*

(i) $p \circ f(t) = t$ *for every* $t \in \mathbb{K}$,

(ii) $f(\mathbb{K})$ *is strongly infinite-dimensional,*

(iii) $M \setminus f(\mathbb{K})$ *is countable-dimensional.*

*In addition, $M$ contains disjoint closed sets $A$ and $B$ such that all partitions in $M$ between $A$ and $B$ meet $f(\mathbb{K})$ in a strongly infinite-dimensional set.*

*Proof:* By Rubin, Schori and Walsh [13], there are a compact space $X$ and a continuous surjection $g$: $X \to \mathbb{K}$ such that for every subset $S \subseteq X$ with $g(S) = \mathbb{K}$ we have that $S$ is strongly infinite-dimensional. By Remark 2.4, there is a first Baire class function $\bar{f}$: $\mathbb{K} \to X$ such that $g \circ \bar{f}(t) = t$ for every $t \in \mathbb{K}$ and $S = \bar{f}(\mathbb{K})$ is a $G_\delta$-subset of $X$. Let $\{(A_i, B_i) : i \in \mathbb{N}\}$ be an essential sequence of pairwise disjoint closed subsets of $S$. Let $\varphi$: $S \to \mathbb{I}$ be a continuous function such that $\varphi(A_1) = 0$ and $\varphi(B_1) = 1$. Since the space $S$ is completely metrizable, it has a compactification $\gamma S$ whose remainder is countable-dimensional; cf. [9, Theorem 3.13.7].

Now consider the product $Y = \gamma S \times \mathbb{K} \times \mathbb{I}$, and the function $\xi$: $S \to Y$ defined by $\xi(s) = (s, g(s), \varphi(s))$. Then $\xi$ is evidently an embedding. Observe that $\xi(S)$ is a closed subspace of $S \times \mathbb{K} \times \mathbb{I}$. As a consequence, the compact space $M = \overline{\xi(S)}$ has the property that $M \setminus \xi(S)$ is contained in $(\gamma S \setminus S) \times \mathbb{K} \times \mathbb{I}$ and hence is countable-dimensional. Now let $p$: $M \to \mathbb{K}$ be the restriction of the projection $Y \to \mathbb{K}$ to $M$, and $f = \xi \circ \bar{f}$. Observe that $\xi(A_1)$ and $\xi(B_1)$ have disjoint closures in $M$. So if $L$ is an arbitrary partition between $A = \overline{\xi(A_1)}$ and $B = \overline{\xi(B_1)}$ in $M$, then $L \cap \xi(S)$ is a partition between $\xi(A_1)$ and $\xi(B_1)$ in $\xi(S)$, and hence is strongly infinite-dimensional by [9, Theorem 3.1.9]. So $M, p$ and $f$ are as required. ∎

## 3. The construction

Throughout this section, $p$: $M \to \mathbb{K}$, $f$: $\mathbb{K} \to M$ and the pair of closed sets $A, B$ in $M$ are as in Proposition 2.8.

By a well-known result of Bing [1] (cf., [9, Theorem 3.8.1]), one can choose a partition $N$ between $A$ and $B$ in $M$ all of whose subcontinua are hereditarily indecomposable, i.e., whenever $C, D$ are continua in $N$ with nonempty intersection, either $C \subseteq D$ or $D \subseteq C$. The set $F = f^{-1}(N)$ is a $G_\delta$-subset of $\mathbb{K}$, $f$ being of the first Baire class. Observe that $f(F) = N \cap f(\mathbb{K})$. Put

$$E = N \cap p^{-1}(f^{-1}(N)) = N \cap p^{-1}(F)$$

and, to simplify the notation, we shall denote by

$$p: E \to F, \quad f: F \to E$$

the restrictions of $p$ and $f$ to $E$ and $F$, respectively. Observe that $p$: $E \to F$ is perfect and surjective. Moreover, $S = f(F)$ is strongly infinite-dimensional, $E \setminus S$ is countable-dimensional, and each continuum in $E$ is hereditarily indecomposable.

By induction, we will construct $G_\delta$-subsets $S_j$ and $H_j$ in $E$, for $j \in \mathbb{N}$, with the following properties:

(1) $S_j$ intersects each fiber of $p$ in exactly one point,
(2) for $t \in F$ the points in $S_j \cap p^{-1}(t)$ and $S \cap p^{-1}(t)$ are joined by a continuum in $p^{-1}(t)$ of diameter at most $1/j$,
(3) $S_j \cap S$ is zero-dimensional,
(4) $S_i \cap S_j \subseteq S$ for $i \neq j$,
(5) $\bigcup_{i \geq j} S_i \subseteq H_j$ and $\bigcap_{j=1}^{\infty} H_j = S$.

We pick for every $j \in \mathbb{N}$ a collection $\mathcal{S}_j$ of closed subsets of $E$ with the properties in Proposition 2.5 for $\varepsilon = 1/j$. We may assume that every element of $\mathcal{S}_{j+1}$ is contained in an element of $\mathcal{S}_j$.

Put $S_0 = S$, and assume that the sets $S_0, \ldots, S_{j-1}$ are defined for certain $j \in \mathbb{N}$. Consider the collection $\mathcal{S}_j$ and fix an element $A \in \mathcal{S}_j$. Applying Proposition 2.7 to the restrictions $p{\restriction}A$: $A \to p(A)$, $f{\restriction}p(A)$: $p(A) \to A$, and $W = (\bigcup_{i<j} S_i) \cap A$, we get a $G_\delta$-set $T_A \subseteq A$ intersecting each fiber of $p{\restriction}A$ in exactly one point, such that

(6) for all $t \in p(A)$, there is a (possibly degenerate) subcontinuum of $p^{-1}(t) \cap A$ that contains the singleton sets $S \cap p^{-1}(t)$ and $T_A \cap p^{-1}(t)$,
(7) if for $t \in p(A)$ the component of $f(t)$ in $p^{-1}(t) \cap A$ is nontrivial, then the unique point contained in $T_A \cap p^{-1}(t)$ does not belong to $S \cup W$,
(8) $T_A \cap S$ is zero-dimensional.

Since the sets $p(A)$, $A \in \mathcal{S}_j$, are closed and pairwise disjoint, and $\mathrm{mesh}(\mathcal{S}_j) \leq 1/j$,

$$S_j = \bigcup \{T_A : A \in \mathcal{S}_j\}$$

is a $G_\delta$-subset of $E$ by Lemma 2.2, and has the properties (1) – (4). To complete the construction, we let

$$H_j = \bigcup \mathcal{S}_j.$$

Then again by Lemma 2.2, it follows that $H_j$ is a $G_\delta$-subset of $E$. Note that $\bigcup_{i \geq j} S_i \subseteq H_j$, since $\mathcal{S}_i$ refines $\mathcal{S}_j$ if $j \leq i$. In addition, we claim that $\bigcap_{j=1}^{\infty} H_j = \bigcap_{j=1}^{\infty} \bigcup \mathcal{S}_j = S$. To see this, first observe that trivially $S \subseteq \bigcap_{j=1}^{\infty} \bigcup \mathcal{S}_j$. Conversely, pick an arbitrary $x \in \bigcap_{j=1}^{\infty} \bigcup \mathcal{S}_j$, and put $x' = f(p(x))$. By (ii) of Proposition 2.5, every element of $\mathcal{S}_j$ that contains $x$ also contains $x'$. Since $\mathrm{mesh}(\mathcal{S}_j) \leq 1/j$ for every $j$, this implies that $x = x'$, i.e., $x \in S$.

We shall justify Examples 1 and 2 by considering the spaces

(8) $X_\triangle = S \cup \bigcup \{S_j : j \in \triangle\}$,

where $\triangle$ is an infinite set of natural numbers.

THEOREM 3.1: *$X_\triangle$ is a complete C-space.*

*Proof:* Since, by (5),

$$X_\triangle = \bigcap_{j=1}^{\infty} (H_j \cup \bigcup \{S_i : i \in \triangle \cap \{1, 2, \ldots, j-1\}\}),$$

the space $X_\triangle$ is a $G_\delta$-subset of $E$, and so it is completely metrizable.

Let us check that $X_\triangle$ is a C-space. We shall first check that each $L \subseteq S$ that is closed in $X_\triangle$ is countable-dimensional. Striving for a contradiction, assume that there is such $L$ that is not countable-dimensional, and put

$$K = \overline{L} \cap p^{-1}(p(L)),$$

where the closure is taken in $E$. Since the restriction $p' = p \restriction K \colon K \to p(K)$ is perfect and $p(K)$ is zero-dimensional, the set of all points in $L$ that belong to a nontrivial continuum in $K$ is not countable-dimensional. This follows easily from the fact that if $x \in K$ is such that the component of $x$ in the (compact) fiber $(p')^{-1}(p(x))$ is trivial, then $K$ is zero-dimensional at $x$. Using this and (2) and (3), one finds a point $x \in L$ and nontrivial continua $C, C_1, C_2, \ldots$, all containing $x$, such that $C \subseteq K$, $\mathrm{diam}\, C_j \leq 1/j$ and $(C_j \cap S_j) \setminus S \neq \emptyset$ for every $j$. Pick $j \in \triangle$ with $1/j < \mathrm{diam}\, C$. Then, the continua in $E$ being hereditarily indecomposable, we must have $C_j \subseteq C$ since $C_j$ and $C$ meet. So there is a point $y \in \overline{L} \cap (X_\triangle \setminus L)$ which contradicts the fact that $L$ is closed in $X_\triangle$.

Now, let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be a sequence of open covers of $X_\triangle$. First write $X_\triangle \setminus S$ as $\bigcup_{n=1}^{\infty} Z_n$, where each $Z_n$ is zero-dimensional. By following the hint in [4, Problem 6.3.D], there is for every $n$ a pairwise disjoint family of open subsets $\mathcal{B}_n$ of $X_\triangle$ covering $Z_n$ and refining $\mathcal{A}_{2n}$. Then $L = X_\triangle \setminus \bigcup_{n=1}^{\infty} \bigcup \mathcal{B}_n$ is contained in $S$, closed in $X_\triangle$, and hence by the above is countable-dimensional. So by the same argument, there is for every $n$ a pairwise disjoint family of open subsets $\mathcal{B}'_n$ of $X_\triangle$ refining $\mathcal{A}_{2n+1}$ such that $L \subseteq \bigcup_{n=1}^{\infty} \bigcup \mathcal{B}'_n$. ∎

## 4. Construction of Example 1

Let $\triangle_0$ and $\triangle_1$ be the sets of even and odd natural numbers, respectively, and let

$$X = X_{\triangle_0} \oplus X_{\triangle_1}$$

be the topological sum of the spaces $X_{\triangle_0}$ and $X_{\triangle_1}$ defined in (8),§3.

Then $X$ is a complete C-space. Since by (4) and (8),§3, $X_{\triangle_0} \cap X_{\triangle_1} = S$, the intersection of $X_{\triangle_0} \times X_{\triangle_1}$ with the diagonal of $X \times X$ can be identified with $S$. Hence $S$ embeds in the square $X \times X$ as a closed subspace, and since $S$ is strongly infinite-dimensional, so is $X \times X$.

## 5. Construction of Example 2

Let us consider the space

$$X_{\mathbb{N}} = S \cup \bigcup_{i=1}^{\infty} S_i.$$

It will be convenient to denote $S$ by $S_0$. By the Cantor–Bendixson Theorem ([3, Problem 1.7.11]), we may assume without loss of generality that $S_0$ is perfect. By a simple transfinite induction, we can partition $S_0$ into a family $\{B_i : i \geq 0\}$ of Bernstein sets, i.e., each $B_i$ intersects every Cantor set in $S_0$. Since $S$ is uncountable-dimensional, we may assume without loss of generality that $B_0$ is uncountable-dimensional as well. Observe that $p(S_i) = F$ for every $i \geq 0$. So we may pick for every $i \geq 1$ a set $T_i \subseteq S_i$ such that $p(T_i) = p(B_i)$, and we put

$$Y = B_0 \cup \bigcup_{i=1}^{\infty} T_i.$$

Observe that $p{\restriction}Y\colon Y \to F$ is a continuous bijection; hence $Y$ is totally disconnected.

As in the proof of Theorem 3.1, we will prove that if $L \subseteq B_0$ is closed in $Y$, then $L$ is countable-dimensional. This will suffice since $Y$ contains $B_0$, which is not countable-dimensional.

Striving for a contradiction, assume that there is such an $L$ that is not countable-dimensional. Let $\overline{L}$ denote the closure of $L$ in $E$. Since the restriction $p\restriction\overline{L}\colon \overline{L} \to p(\overline{L})$ is perfect and $p(\overline{L})$ is zero-dimensional, the set $A$ of all points in $\overline{L}$ that do not belong to a nontrivial continuum in $\overline{L}$ is a zero-dimensional $G_\delta$-subset of $\overline{L}$. Indeed, let us consider the monotone-light factorization $p\restriction\overline{L} = u \circ v$ described in (3), (4) in §2, with $E = \overline{L}$, $F = p(\overline{L})$. By Lemma 2.3, $A$ is a $G_\delta$-set in $\overline{L}$. Since $u$ is monotone, $u^{-1}(u(a)) = \{a\}$ for $a \in A$, and hence $u$ embeds $A$ into a zero-dimensional space; cf. (5) in §2.

Hence $L \setminus (A \cup \bigcup_{j=1}^{\infty} S_j)$ is uncountable-dimensional, by (3) in §3. For every $x \in \overline{L}$ let $C_x$ be the component of $x$ in $\overline{L}$. Observe that $C_x$ is a continuum in $p^{-1}p(x)$, since $p\restriction\overline{L}\colon \overline{L} \to p(\overline{L})$ is perfect and $p(\overline{L})$ is zero-dimensional. For every $x \in L \setminus (A \cup \bigcup_{j=1}^{\infty} S_j)$ we have that $\operatorname{diam} C_x > 0$. Hence, $L \setminus (A \cup \bigcup_{j=1}^{\infty} S_j)$ being uncountable, there are an uncountable subset $Z$ of $L \setminus (A \cup \bigcup_{j=1}^{\infty} S_j)$ and an $\varepsilon > 0$ such that for every $x \in Z$, $\operatorname{diam} C_x \geq \varepsilon$.

By Lemma 2.3, also for each $x \in \overline{Z}$, $\operatorname{diam} C_x \geq \varepsilon$. Since $(\overline{Z} \cap S) \setminus (A \cup \bigcup_{j=1}^{\infty} S_j)$ is a Borel set containing $Z$, and hence uncountable, it contains a Cantor set $T$. In effect, we get a Cantor set $T \subseteq \overline{L} \cap S$ with $\operatorname{diam} C_x \geq \varepsilon$ for $x \in T$.

Pick $j$ so that $1/j < \varepsilon$. The Cantor set $T$ intersects $B_j$, and let $x$ belong to this intersection. Since $x \in B_j$, there is $y \in T_j \subseteq S_j$ with $p(y) = p(x)$. By (2) in§3 there is a continuum $C$ in $p^{-1}p(x)$ joining $x$ and $y$ with $\operatorname{diam} C \leq 1/j$. The continua in $E$ being hereditarily indecomposable, we must have $C \subseteq C_x$, since $C$ and $C_x$ meet. So $y \in \overline{L} \cap (Y \setminus L)$, which contradicts the fact that $L$ is closed in $Y$.

## References


[1] R. H. Bing, *Higher-dimensional hereditarily indecomposable continua,* Transactions of the American Mathematical Society **71** (1951), 267–273.

[2] N. Bourbaki, *Topologie Générale* (2e ed.), Hermann, Paris, 1958.

[3] R. Engelking, *General Topology,* second ed., Heldermann Verlag, Berlin, 1989.

[4] R. Engelking, *Theory of Dimensions, Finite and Infinite,* Heldermann Verlag, Berlin, 1995.

[5] V. V. Fedorchuk and J. van Mill, *Dimensionsgrad for locally connected Polish spaces,* Fundamenta Mathematicae **163** (2000), 77–82.

[6] J. G. Hocking and G. S. Young, *Topology,* Dover, New York, 1988.

[7] A. S. Kechris, *Classical Descriptive Set Theory,* Springer-Verlag, New York, 1995.

[8] K. Kuratowski, *Topology,* Vols. I, II, Academic Press, New York 1966, 1968.

[9] J. van Mill, *The Infinite-dimensional Topology of Function Spaces*, North-Holland, Amsterdam, 2001.

[10] J. van Mill and R. Pol, *On spaces without non-trivial subcontinua and the dimension of their products*, Topology and its Applications **142** (2004), 31–48.

[11] E. Pol, *A weakly infinite-dimensional space whose product with the irrationals is strongly infinite-dimensional*, Proceedings of the American Mathematical Society **98** (1986), 349–352.

[12] E. Pol, *Spaces whose $n$th power is weakly infinite-dimensional, but whose $(n+1)$th power is not*, Proceedings of the American Mathematical Society **117** (1993), 871–876.

[13] L. Rubin, R. M. Schori and J. J. Walsh, *New dimension-theory techniques for constructing infinite-dimensional examples*, General Topology and its Applications **10** (1979), 93–102.